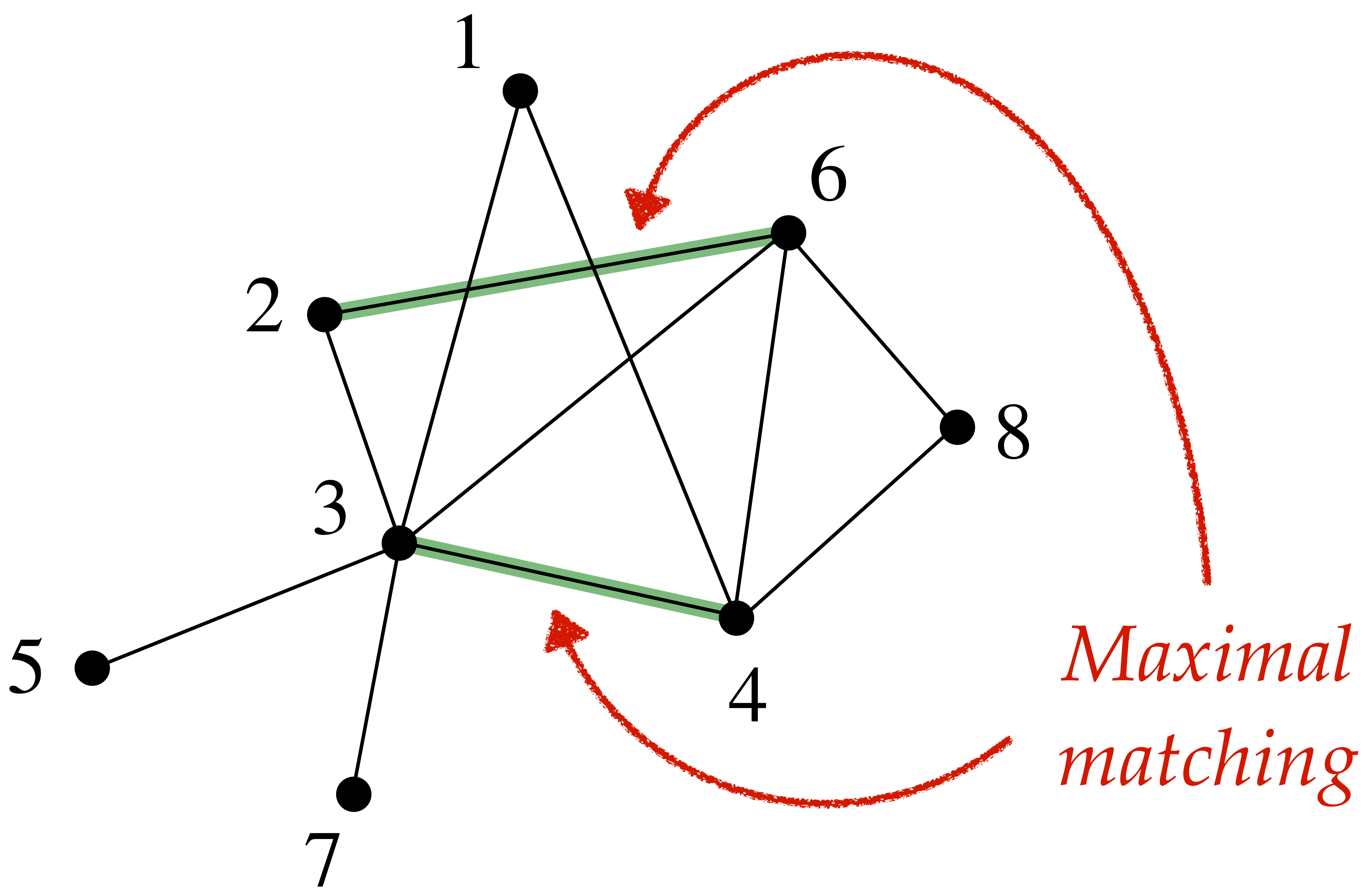
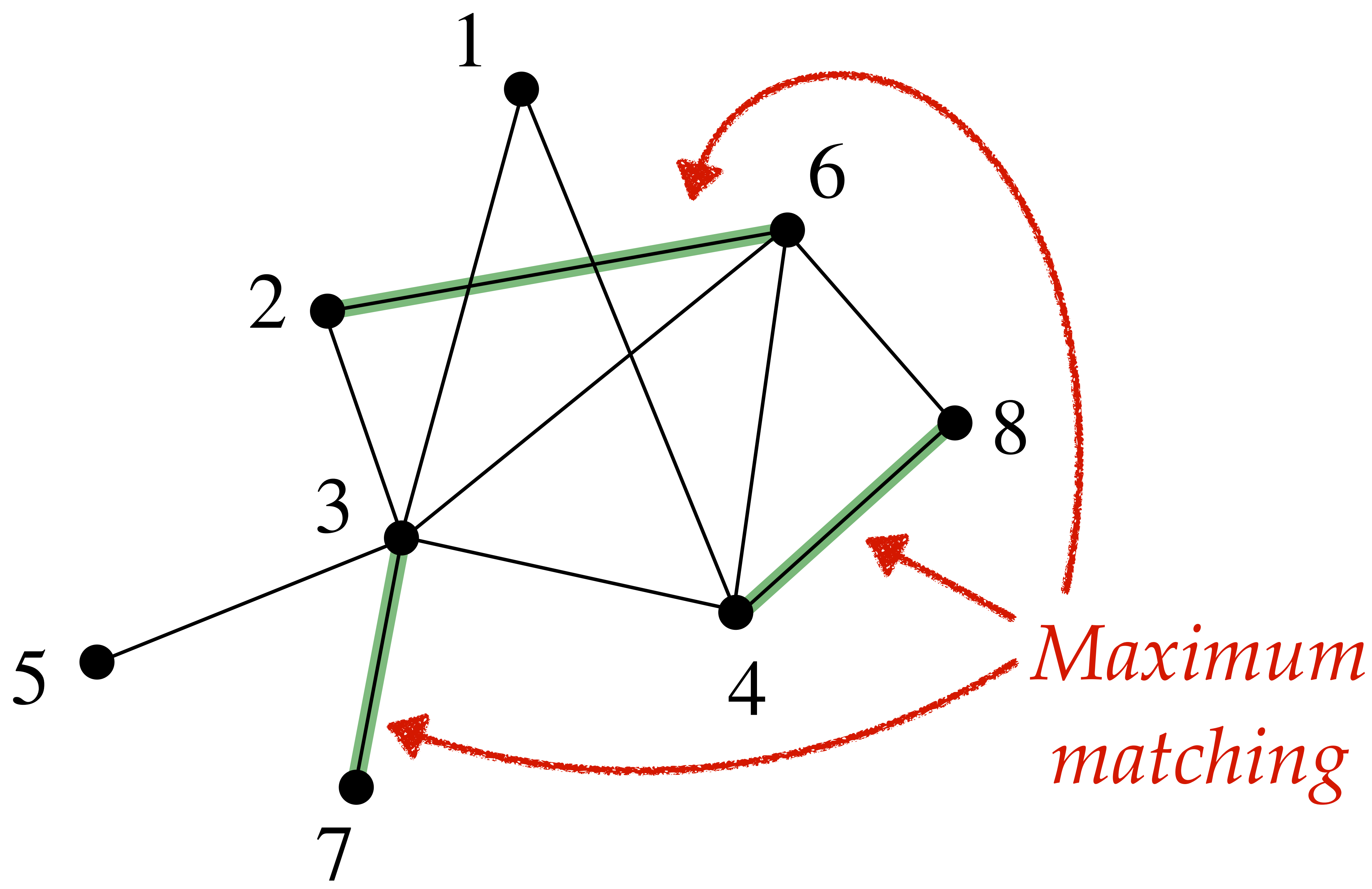
Lecture 32

Matching

Matching

Definition: A **matching** in a graph $G = (V, E)$ is a subset $M \subseteq E$ so that no two edges in M are incident with a common vertex.



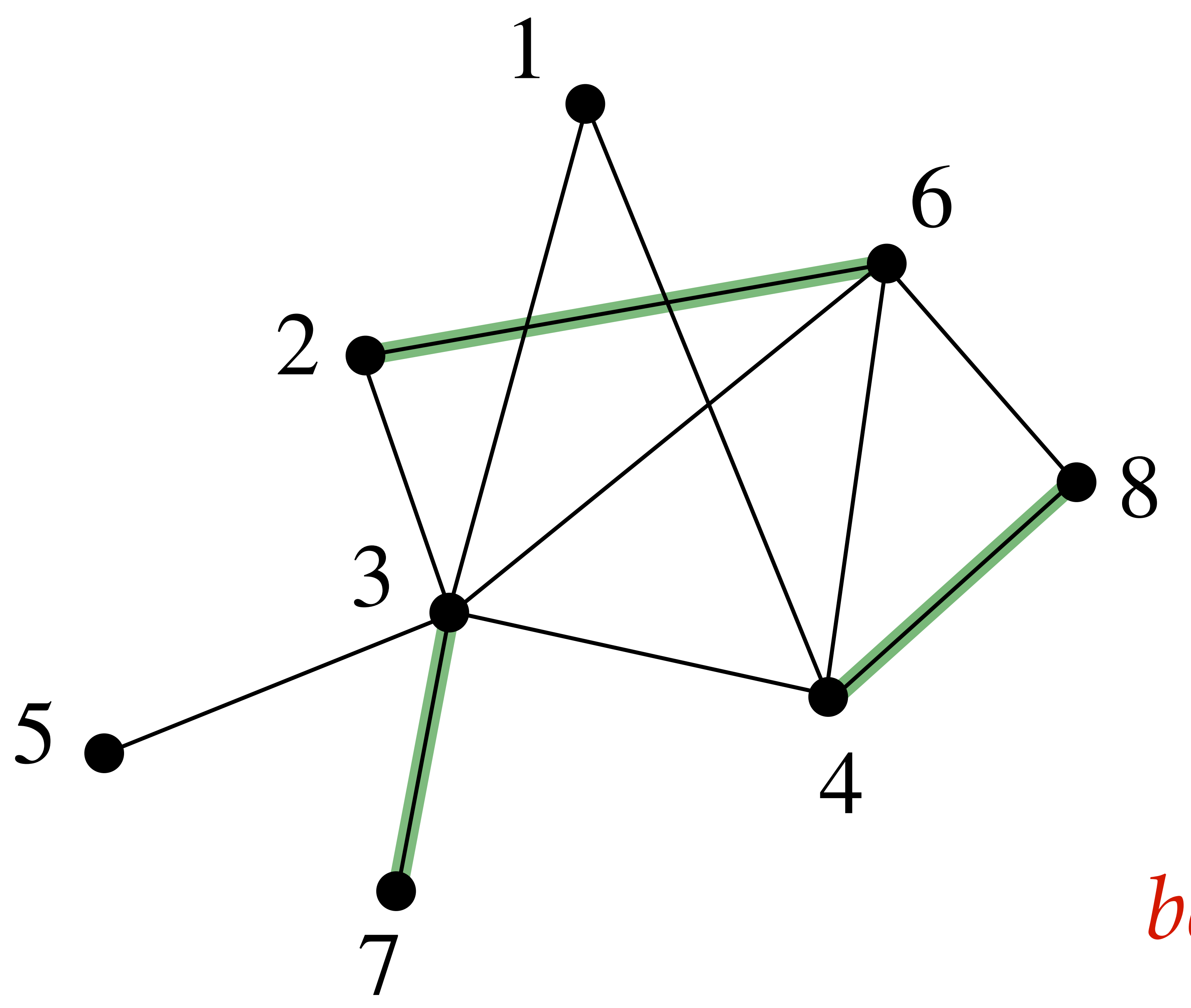


Definition: A matching in M is **maximal** if there is no matching M' with $M \subset M'$ and is **maximum** if there is no matching M'' such that $|M| < |M''|$.

Cover and Perfect Matching

Definition: We say a matching M **covers** a set of vertices of vertices X, if every $x \in X$ is incident with some edge in M.

M —



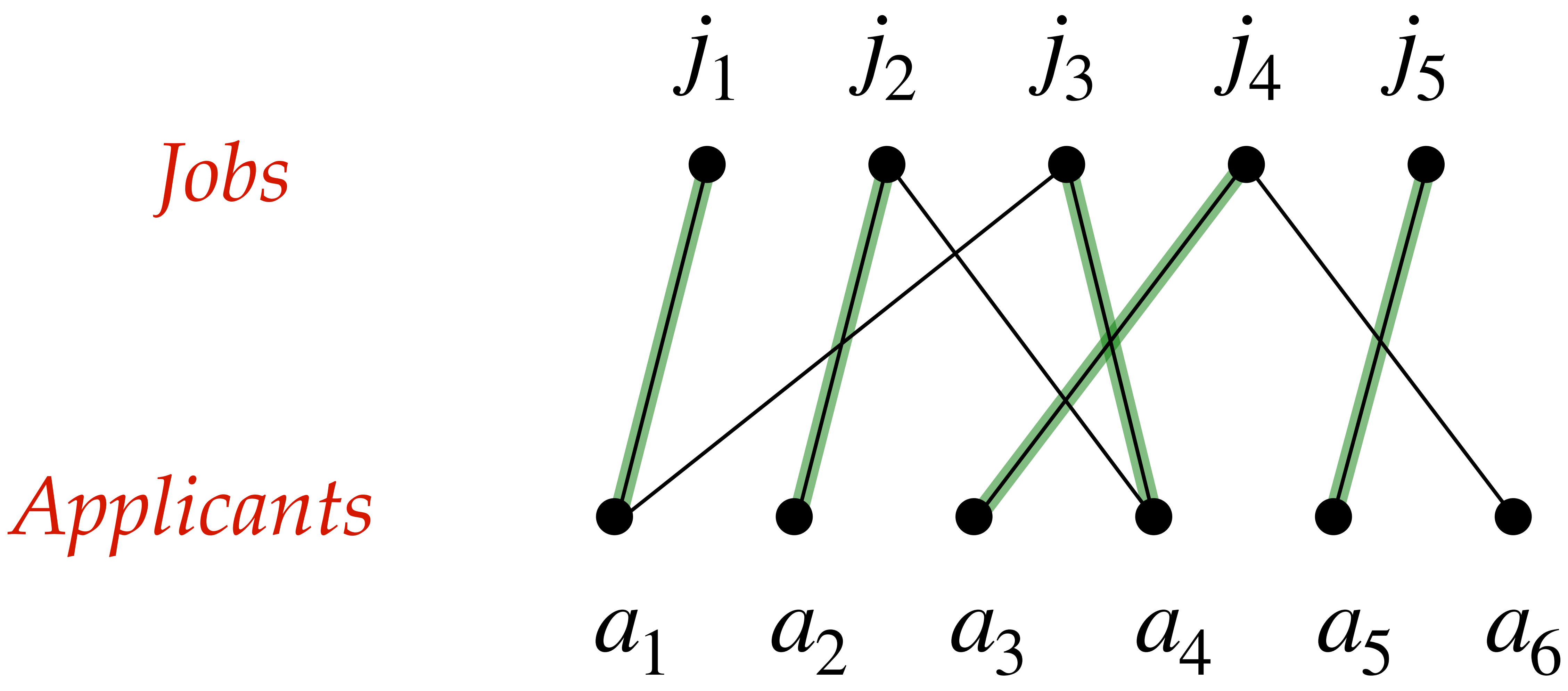
M covers $\{2,6,3,7,4,8\}$

Perfect matching in this graph is not possible because we can either cover 5 or 7, but not both.

Definition: We say a matching in a graph G is **perfect** if it covers all the vertices of G.

Matching in Real Life

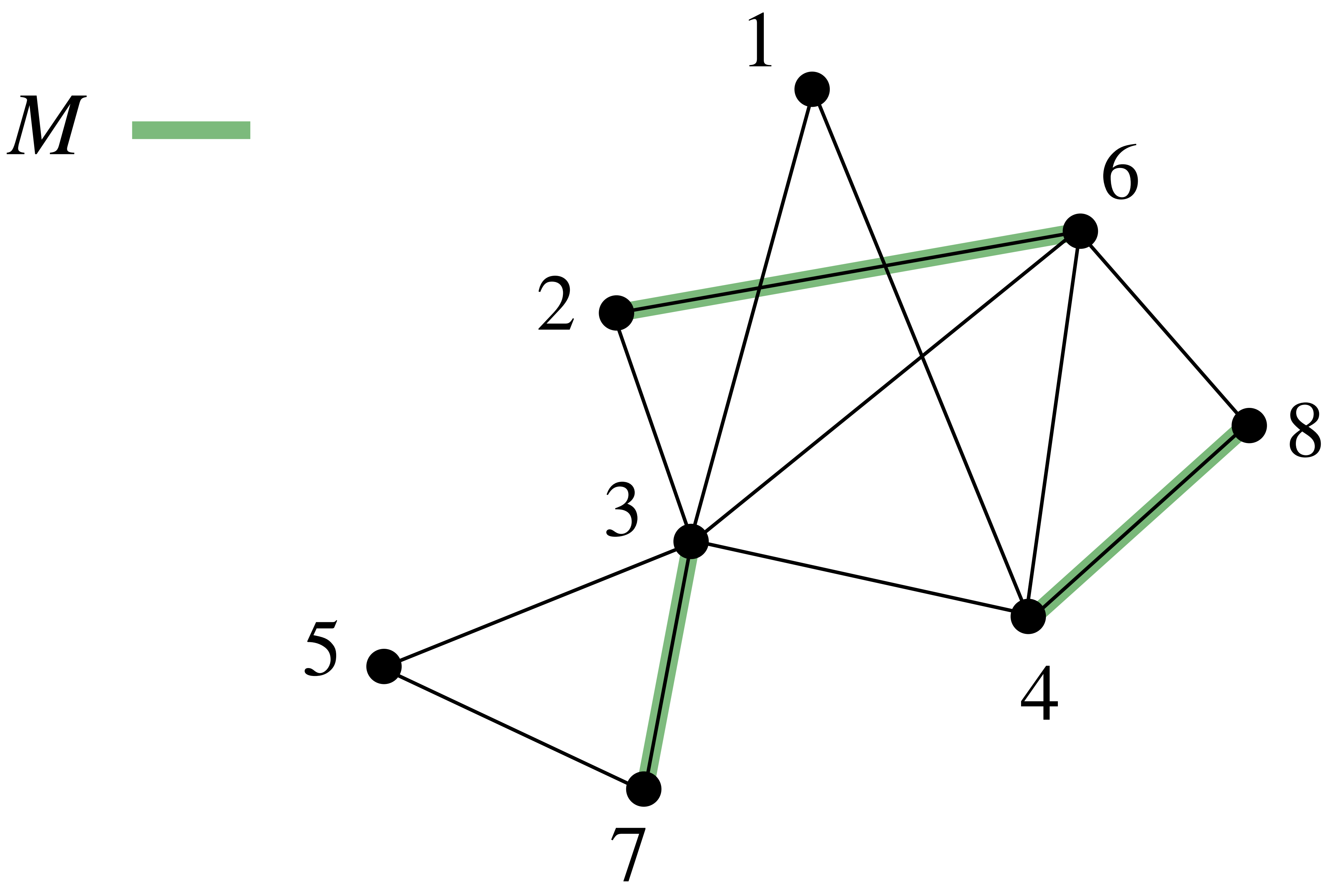
Suppose there are 5 job openings and 6 applicants. We want to fill each job opening by hiring exactly one applicant and one applicant can do at most one job.



Observation: All the jobs can be filled if and only if there exists a matching M that covers $\{j_1, j_2, j_3, j_4, j_5\}$.

Alternating & Augmenting Path

Definition: If M is a matching in $G = (V, E)$, a path P in G is **M-alternating** if the edges of P belong alternately to M and to $E \backslash M$. A path P is **M-augmenting** if P is M-alternating and its distinct end points u and v are not incident with an edge or edges of M.



Some M-alternating paths are:

$\langle 1,4,8,6,2,3,7 \rangle$, $\langle 7,3,2,6 \rangle$, $\langle 4,8 \rangle$, $\langle 1,3 \rangle$

Some M-augmenting paths are:

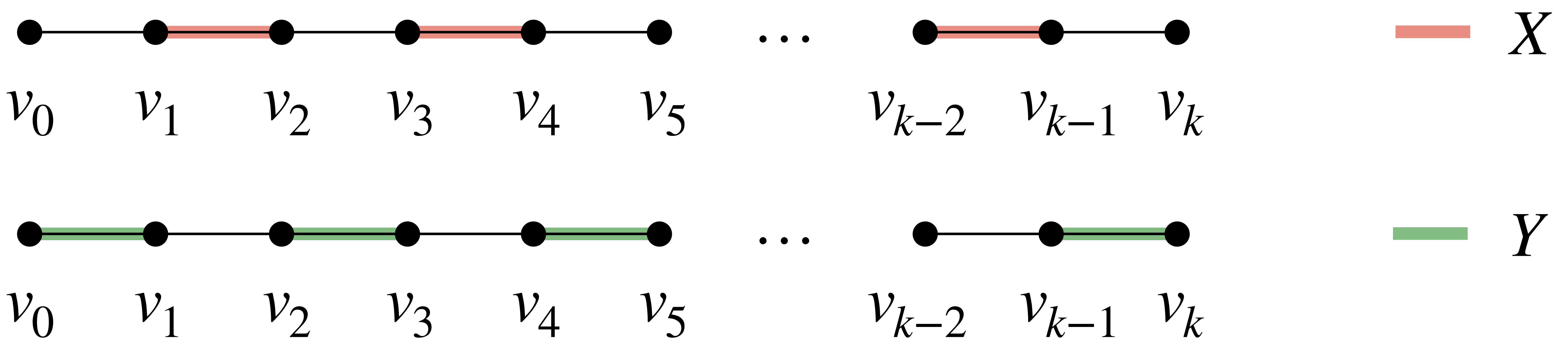
$\langle 1,4,8,6,2,3,7,5 \rangle$, $\langle 5,7,3,1 \rangle$

Berge's Theorem

Theorem: A matching M is maximum if and only if there is no M-augmenting path.

Proof: ($\implies$) If there is an M-augmenting path, then M is not a maximum matching.

Let P be an M-augmenting path.



Let X be the set of edges in P that are in M and let Y be the rest of the edges.

Then $(M \backslash X) \cup Y$ will a be a larger matching than M.

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